With (5), (34) can be written in conventional units

$$n\,k\,T \ll \varepsilon_0\,E^2 \quad (\text{VA sec/cm}^3) \qquad (35)$$

i. e. independent of x, in every space point, the thermal energy density $\frac{3}{2}\,n\,k\,T$ must be negligibly small compared to the electrostatic field energy density $\frac{1}{2}\,\varepsilon_0\,E^2$.

With $k = 1.38 \cdot 10^{-23}$ (VA sec/grad), $\varepsilon_0 = 0.885 \cdot 10^{-13}$ (A sec/V cm)

(5) becomes

$$n\,T \ll 10^{10}\,E^2\,. \qquad (36)$$

For a space charge formula as (30), which postulates $E_0 = 0$, the charge carriers at $x = 0$ should be emitted with $T = 0$.

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The Propagation of Periodic Waves in a Two-level System

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In this article the possibility for the propagation of periodic waves in a two-level system is discussed. It turns out, that in the absence of processes leading to loss and gain of energy, respectively, periodic waves only are possible. Their frequencies are always greater than the transition frequency of the two-level system. The frequency difference and the deviation of the wave profile from a sinusoidal form increases with the strength of interaction of light field and two-level systems.

After the introduction of loss and source terms it turns out that only isolated periodic waves exist. The asymptotically stable ones among them act as asymptotic solutions to all other wave solutions. The frequencies and wave forms are affected in a similar way as without loss and source terms.

1. Introduction

When by the development of the laser coherent light sources of high intensity became available, the problem of propagation of coherent light waves in a medium, absorbing near the frequency of the light wave, became of considerable interest[1−9]. An excellent and exhausting survey on the state of affairs has recently been published by Arecchi and his coworkers[10].

Usually, the starting point is the assumption that the absorbing medium can be replaced by a set of two-level systems with nondegenerate energy levels The evolution of the quantities characterizing the state of the medium, i. e. inversion and polarization, is treated quantum mechanically, whereas the propagation of the light wave and its interaction with the material system is described in a purely classical way (semi-classical theory)[11]. One of the most important questions posed in this context is whether such a system admits wave solutions, i. e. solutions which do not change their form during propagation.

The method which is followed for the investigation of this problem consists in assuming for the

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electric field strength a solution of the following form

$$E(x,t) = E_0(x,t)\cos\left[\frac{\omega_0}{c}(x+ct) + \Phi(x,t)\right] \quad (1.1)$$

and similarly for the inversion and the polarization of the medium. These expressions are inserted into the equations of motion and terms changing with double frequency are neglected. Furthermore, it is assumed[12] that E_0 and Φ are slowly varying functions compared with the sinusoidal basic wave:

$$\left|\frac{\partial E_0}{\partial x}\right| \ll \left|\frac{E_0}{c/\omega_0}\right|; \quad \left|\frac{\partial E_0}{\partial t}\right| \ll |\omega_0\, E_0|; \quad 1.2)$$

$$|\partial\Phi/\partial x| \ll \omega_0/c; \quad |\partial\Phi/\partial t| \ll \omega_0. \quad (1.3)$$

Fundamental for the hypothesis (1.1) are the assumptions that a) the frequency and the phase velocity of the carrier wave are not affected by the strength of interaction between light wave and material system and that b) the carrier wave is sinusoidal. It is the purpose of this article to throw some light on these questions. For this we will be looking for periodic solutions of the corresponding system of differential equations without making use of the hypothesis (1.1). It will turn out that with increasing strength of interaction between light field and medium deviations of the wave profile from sinusoidal form occur and the frequency increases. From the simple model we are considering here no restriction results, however, concerning the phase velocity.

2. The Model Equations

We assume a set of two-level systems which do not interact with each other, but are exposed to a perturbation by a light wave. The state of such a system is given by

$$\psi(t) = c_1(t)\,\psi_1 + c_2(t)\,\psi_2, \quad (2.1)$$

where the wave functions corresponding to the two energy levels are denoted by ψ_1 and ψ_2, respectively, with eigenvalues ε_1 and ε_2. Restricting ourselves to dipole interaction with the light field $E(t)$, we obtain in the usual way from the Schrödinger equation the evolution equations for the amplitudes c_1 and c_2:

$$i\hbar\dot{c}_1 = \varepsilon_1 c_1 - e x_{12} E(t)\, c_2, \quad (2.2)$$

$$i\hbar\dot{c}_2 = \varepsilon_2 c_2 - e x_{12} E(t)\, c_1. \quad (2.3)$$

The transition moment x_{12} is assumed to be real. Using (2.2) and (2.3), we can write down the equations of motion for the four real basic combinations of the amplitudes. The first can be integrated immediately and expresses the conservation of the total probability:

$$c_1^* c_1 + c_2^* c_2 = N_0 \quad (2.4)$$

where N_0 is the number of two-level systems per unit volume. The remaining three lead to the following system of differential equations

$$\frac{\partial C}{\partial t} = \omega_0 S \quad (2.5)$$

$$\frac{\partial S}{\partial t} = -\,\omega_0 C - \frac{2}{\hbar} e^2 x_{12}^2 E\sigma \quad (2.6)$$

$$\frac{\partial \sigma}{\partial t} = \frac{2}{\hbar} E S. \quad (2.7)$$

The dependent variables

$$C = N_0 e x_{12}(c_1 c_2^* + c_1^* c_2), \quad (2.8)$$

$$S = i N_0 e x_{12}(c_1 c_2^* - c_1^* c_2), \quad (2.9)$$

$$\sigma = N_0(c_2^* c_2 - c_1^* c_1) \quad (2.10)$$

are the „in-phase" component of the polarization, the "in-quadrature" component and the inversion, respectively.

$$\omega_0 = \frac{1}{\hbar}(\varepsilon_2 - \varepsilon_1) \quad (2.11)$$

is the frequency of the transition between ψ_1 and ψ_2. To these equations for the material system is added the wave equation for the electric field strength

$$c^2\frac{\partial^2 E}{\partial x^2} - \frac{\partial^2 E}{\partial t^2} - 4\pi\frac{\partial^2 C}{\partial t^2} = 0. \quad (2.12)$$

In the following we are interested in running wave solutions only, i. e. we are looking for solutions which depend only on the combination

$$\tau = \pm x + v t \quad (2.13)$$

of the two independent variables with v as arbitrary phase velocity. Denoting the differentiation with respect to τ by $\mathrm{d}/\mathrm{d}\tau\ (\cdot) = (\cdot)'$, we obtain instead of system (2.5), (2.6), (2.7), (2.12) of partial differential equations a system of ordinary differential equations:

$$(c^2 - v^2)E'' - 4\pi v^2 C'' = 0, \quad (2.14)$$

$$C' = (\omega_0/v)\,S, \quad (2.15)$$

[12] An exception is Ref. 6.

$$S' = -\frac{\omega_0}{v} C - \frac{2}{\hbar v} e^2 x_{12}^2 E \sigma, \quad (2.16)$$

$$\sigma' = (2/\hbar v) ES. \quad (2.17)$$

We note that with the reduction to system (2.14) —(2.17) we have renounced to learn something about the dependence of the phase velocity on the intensity of the light field, since we have treated v as a constant. Otherwise, the resulting system would have presented itself in a much more complicated form.

The integration of (2.14) yields (for $v \neq c$):

$$C = \frac{1}{4\pi} \left(\frac{c^2}{v^2} - 1\right) E \quad (2.18)$$

a well known relation in electrodynamics. A consequence of the assumption of a constant phase velocity is, therefore, that the range of validity of our considerations is given by the range of validity for the proportionality of polarisation and electric field strength.

Using relations (2.18) and (2.15), we can eliminate C and S and, subsequently, integrate (2.17):

$$\sigma = \frac{1}{\hbar \omega_0} \frac{c^2 - v^2}{4\pi v^2} E^2. \quad (2.19)$$

Finally, the elimination of σ leads to

$$E'' + \frac{\omega_0^2}{v^2} E + 2 \frac{e^2 x_{12}^2}{\hbar^2 v^2} E^3 = 0. \quad (2.20)$$

3. Discussion of Solutions

Eq. (2.20) is a Duffing equation which can be solved exactly. The solution reads

$$E(\tau) = E_m \, cn \left[\frac{\tau}{\hbar v} (\hbar^2 \omega_0^2 + 2 e^2 x_{12}^2 E_m^2)^{1/2}; \left(\frac{e^2 x_{12}^2 E_m^2}{\hbar^2 \omega_0^2 + 2 e^2 x_{12}^2 E_m^2}\right)^{1/2}\right], \quad (3.1)$$

which can be proved by differentiating (3.1) and inserting the result into (2.20). $cn(x; k)$ is the elliptic cosine with argument x and module k. E_m is a integration constant, the second has been chosen in a way as to make E assume its maximal value at $\tau = 0$. For $v = c$, however, Eq. (2.20) is not equivalent to the system (2.14)—(2.17). Therefore, we have to return to the latter. If we admit only bounded solutions, the integration of (2.14) yields $C = 0$ and we obtain the total solution for (2.14) —(2.17):

$$C = 0; \quad S = 0; \quad \sigma = 0; \quad E = f(\tau) \quad (3.2)$$

with f arbitrary.

It is interesting to compare the solutions of the wave equation coupled to a resonant medium with the solutions of the simple wave equation:

$$c^2 (\partial^2 E / \partial x^2) - (\partial^2 E / \partial t^2 = 0. \quad (3.3)$$

The classic solution of (3.3) for running waves is

$$E = f(\pm x + c t) \quad (3.4)$$

with arbitrary f. This means: All waves propagate with the phase velocity c. For $v \neq c$ no bounded solutions exist except the zero solution. The wave profile f is arbitrary, it has only to be twice differentiable with respect to both independent variables.

In the case of an electromagnetic wave coupled to a resonant medium it results that solutions different from zero exist for all velocities $v \neq c$. The wave profile, however, has to be periodic. The propagation of free electromagnetic waves and of electromagnetic waves in a resonant medium are, in a certain sense, complementary.

A further difference is that for a light wave propagating in a resonant medium the frequency is no more independent of the intensity. Making use of the fact that $c(x; k)$ has a period of $4K(k)$, where $K(k)$ is the complete elliptic integral of the first kind[13], we derive from (3.1) that $E(\tau)$ has a period T which is determined by

$$\frac{T}{\hbar v} (\hbar^2 \omega_0^2 + 2 e^2 x_{12}^2 E_m^2)^{1/2} = 4K \left[\left(\frac{e^2 x_{12}^2 E_m^2}{\hbar^2 \omega_0^2 + 2 e^2 x_{12}^2 E_m^2}\right)^{1/2}\right]. \quad (3.5)$$

Taking into account also (2.13), we obtain for the frequency ω

$$\omega = \frac{\pi}{2} \hbar^{-1} (\hbar^2 \omega_0^2 + 2 e^2 x_{12}^2 E_m^2)^{1/2} K^{-1}. \quad (3.6)$$

After some manipulation results, finally,

$$\frac{\omega}{\omega_0} = \frac{\pi}{2} \left(1 + 2 \frac{e^2 x_{12}^2 E_m^2}{\hbar^2 \omega_0^2}\right)^{1/2} \cdot K^{-1} \left[\left(\frac{e^2 x_{12}^2 E_m^2}{\hbar^2 \omega_0^2 + 2 e^2 x_{12}^2 E_m^2}\right)^{1/2}\right]. \quad (3.7)$$

Consequently, the ratio ω/ω_0 depends only on the ratio of the interaction energy of the polarizable medium in the electromagnetic field and the energy difference of the transition. This dependence is represented in Fig. 1. The frequency of the periodic wave is always greater than the frequency of the

[13] See e. g. JAHNKE, E. and F. EMDE, Tables of Functions, Dover Publications, New York 1945.

transition and increases monotonically with the intensity of the electromagnetic field.

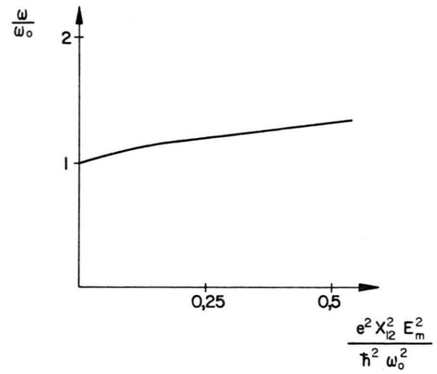

Fig. 1. Dependence of the frequency ω on the ratio of the interaction energy with the light field and the energy difference of the two-level system.

4. Extension of the Model

The differential Eq. (2.20) has a first time-independent integral

$$(E')^2 + \frac{\omega_0^2}{v^2} E^2 + \frac{e^2 x_{12}^2}{\hbar^2 v^2} E^4 = \text{const}. \qquad (4.1)$$

It is just this conservation law which permits only periodic solutions to (2.20), i.e. which gives to our problem the character of a center. Such a problem, however, is structurally unstable in the sense that arbitrary small perturbations of the differential equation change the qualitative character of the solutions. Since in every physical system losses are inevitable, e.g. in our case by destruction of the coherence of polarization or by weakening of the electromagnetic field in consequence of irreversible processes (electric resistance), such a theory is not very satisfactory.

The most natural way to introduce losses and pumping processes necessary for their compensation would be to add corresponding terms in system (2.14)—(2.17). An attempt, however, shows immediately that then a reduction to an autonomous differential equation, corresponding to (2.20), is no more possible. Since we are interested here mainly in a qualitative analysis, it seems justified to us to start instead from Eq. (2.20) in order to render the calculations and their interpretation more transparent. The method employed in the following is applicable also to an extended system (2.14)—(2.17).

It is evident that in an extended Eq. (2.20) gain and loss terms cannot be functions of E alone, but must contain E'. For every function $\gamma \psi(E)$ with $\psi(0) = 0$, added to (2.20), leads to an additional term $\gamma \int_0^E \psi(E) \, dE$ in (4.1), which does not change the centrum character of the problem[14] for sufficiently small $\gamma > 0$.

As loss term we will take $\varkappa / v E'$, where $\varkappa$ is e.g. a non vanishing conductivity. Since we conceive the source term as being brought about by pumping of the resonant transition of the medium and subsequent induced emission into the field, we will assume the source term for small intensities as proportional to the intensity of the light field. On the other hand, as the inversion has a finite maximal value, the efficiency of the source must tend towards zero for the intensity tending towards infinity. Thus, we will assume the source term in the form

$$\text{source term} = \frac{P}{v} E' g(E^2) \qquad (4.2)$$

with

$$g(E^2) > 0 \quad \text{for} \quad E^2 \neq 0, \qquad (4.3)$$
$$g(0) = 0; \quad \lim_{E^2 \to \infty} g(E^2) = 0.$$

p is a parameter, measuring the strength of the source (pump parameter).

The extended differential equation for the propagation of electromagnetic waves in a resonant medium with losses and sources will be assumed in the form

$$E'' + \frac{1}{v} (\varkappa - p g(E^2)) E' + \frac{1}{v^2} \left(\omega_0^2 E + 2 \frac{e^2 x_{12}^2}{\hbar^2} E^3 \right) = 0. \qquad (4.4)$$

It is no more possible, of course, to integrate, in general, this equation in closed form and we have to apply other methods for the investigation of its solutions. We make the assumption that all processes, which are responsible for losses and gain, are slow compared with ω_0^{-1}. Furthermore, the interaction energy of the polarization and the electric field is assumed to be small in comparison with $\hbar \omega_0$. In order to express this we introduce a parameter μ, which we will use as an ordering parameter in a perturbation expansion and which we will put eventually equal to 1. Moreover, we intro-

[14] If $\psi(E) \geq 0$, then the problem retains its centrum character even for arbitrary $\gamma > 0$.

duce the new independent variable

$$\tau' = (\omega_0/v)\,\tau\,. \quad (4.5)$$

Denoting the differentiation with respect to the new variable again by $(\cdot)'$, we may write instead of (4.4)

$$E'' + E = \mu F(E, E') \quad (4.6)$$

with

$$F(E, E') \equiv -2\,\frac{e^2 x_{12}^2}{\hbar^2 \omega_0^2} E^3 - \frac{\varkappa}{\omega_0} E' + \frac{P}{\omega_0} E' g(E^2)\,. \quad (4.7)$$

The differential Eq. (4.6) can be transformed in an equivalent system of integral equations:

$$E(\tau', \mu) = b(\mu)\sin\tau' + \mu \int_0^{\tau'} \sin(\tau' - s)\, F(E(s), E'(s))\, ds\,, \quad (4.8)$$

$$E'(\tau', \mu) = b(\mu)\cos\tau' + \mu \int_0^{\tau'} \cos(\tau' - s)\, F(E(s), E'(s))\, ds \quad (4.9)$$

with the initial conditons

$$E(0, \mu) = 0; \quad E'(0, \mu) = b(\mu)\,. \quad (4.10)$$

By differentiation of (4.9) and insertion into (4.6) it can easily be proved that (4.8) and (4.9) satisfy, indeed, the differential Eq. (4.6).

5. Periodic Solutions

Before we discuss the behavior of solutions to (4.6) in general, we will search for periodic solutions. For $\mu = 0$ there exists a family of periodic solutions, all with period 2π and covering densely the whole phase plane. This is no longer true for $\mu \neq 0$, however small. It arises the question, whether there are periodic solutions for $\mu = 0$, which remain periodic for $\mu \neq 0$ continuously increasing from zero.

A necessary and sufficient condition for this can be give nby means of the integral Eqs. (4.8) and (4.9). We express the period of the sought periodic solution in form of a power series with respect to μ

$$T' = 2\pi + \eta(\mu) = 2\pi + \eta_1\mu + \eta_2\mu^2 + \cdots \quad (5.1)$$

The periodicity condition may be written as

$$E(T', \mu) = E(0, \mu); \quad E'(T', \mu) = E'(0, \mu)\,, \quad (5.2)$$

which in view of (5.1) leads to

$$A(b, \eta) \equiv b\sin\eta + \mu \int_0^{T'} \sin(\eta - s)\, F(E, E')\, ds = 0 \quad (5.3)$$

and

$$B(b, \eta) \equiv b(\cos\eta - 1) + \mu \int_0^{T'} \cos(\eta - s)\, F(E, E')\, ds = 0\,. \quad (5.4)$$

When we expand b, too, in power series with respect to μ, A and B become functions of the single variable μ. It is our goal to expand also A and B in a Taylor series with respect to μ. Since the conditions (5.3) and (5.4) must be valid for every sufficiently small μ and, therefore, for every single term in the corresponding expansions, we obtain a system of algebraic equations which we will use for the determination of the expansion coefficients $b_0, b_1, \ldots, \eta_1, \eta_2, \ldots$.

After a somewhat cumbersome, although not difficult calculation, we obtain the Taylor expansion

$$A = A_1\mu + A_2\mu^2 + \cdots = 0\,, \quad (5.6)$$

$$B = B_1\mu + B_2\mu^2 + \cdots = 0\,, \quad (5.7)$$

where

$$A_1 = b_0\eta_1 - \int_0^{2\pi} \sin s\, F(b_0 \sin s, b_0 \cos s)\, ds\,, \quad (5.8)$$

$$B_1 = \int_0^{2\pi} \cos s\, F(b_0 \sin s, b_0 \cos s)\, ds\,, \quad (5.9)$$

$$A_2 = b_0\eta_2 + b_1\eta_1 + \eta_1 B_1 - b_1 \int_0^{2\pi} (\sin^2 s\, F_E[s] + \sin s \cos s\, F_{E'}[s])\, ds - \int_0^{2\pi} \sin s\, F_E[s] \int_0^{s} \sin(s - s')\, F[s']\, ds'\, ds\,, \quad (5.10)$$

$$B_2 = \tfrac{1}{2} b_0 \eta_1^2 - A_1\eta_1 + b_1 \int_0^{2\pi} (\sin s \cos s\, F_E[s] + \cos^2 s\, F_{E'}[s])\, ds + \int_0^{2\pi} \cos s \Big[F_E[s] \int_0^{s} \sin(s - s')\, F[s']\, ds' + F_{E'}[s] \int_0^{s} \cos(s - s')\, F[s']\, ds'\Big] ds\,. \quad (5.11)$$

$F_E[s]$ and $F_{E'}[s]$ are here abbreviations for

$$F_E[s] = \frac{\partial}{\partial E} F(E, E')\bigg|_{E = b_0 \sin s;\; E' = b_0 \cos s}, \quad (5.12)$$

$$F_{E'}[s] = \frac{\partial}{\partial E'} F(E, E')\bigg|_{E = b_0 \sin s;\; E' = b_0 \cos s.} \quad (1.35)$$

The periodicity conditions (5.3) and (5.4) in view of the reason, mentioned above, lead to a system of conditions $A_1=0$, $B_1=0$, $B_2=0, \dots$. If and only if this system is solvable, periodic solutions to (4.6) exist.

First, let us consider the condition $B_1=0$. Inserting (4.7) into (5.9), we obtain after some manipulation

$$\left\{\int_0^{2\pi} \cos^2 s\, g(b_0^2 \sin^2 s)\, \mathrm{d}s - \frac{\varkappa\pi}{P}\right\} b_0 = 0. \quad (5.14)$$

This is an equation for the determination of b_0. It may be seen immediately by inspection of the Eqs. (5.8), (5.10) and (5.11) that for every solution $b_0^{(i)}$ of (5.14) the other coefficients η_1, b_1, $\eta_2, \dots$ can be calculated easily.

(5.14) has the trivial solution $b_0=0$, which leads at once to $\eta_1=0$, $b_1=0$, $\eta_2=0, \dots$: the zero solution is a trivial periodic solution without interest to us. Let us now set to zero the expression in brackets in (5.14). It is evident that with every b_0 also $-b_0$ is a solution. Since such a change of sign is equivalent to a shift of the origin of the variable τ' by half a period, we may suppose b_0 to be positive without restriction of generality. Taking into account (4.3), we conclude that the integral in (5.14) is positive for all b_0, tends to zero for $b_0 \to \infty$ and vanishes for $b_0=0$:

$$I(b_0^2) \equiv \int_0^{2\pi} \cos^2 s\, g(b_0^2 \sin^2 s)\, \mathrm{d}s > 0 \quad \text{for} \quad b_0^2 \neq 0;$$
$$I(0)=0; \quad I(\infty)=0. \quad (5.15)$$

On the whole, $I(b_0^2)$ reflects the behavior of $g(b_0^2)$. As is clear from Fig. 2, for p sufficiently small (5.14) has no solution different from zero. For p passing

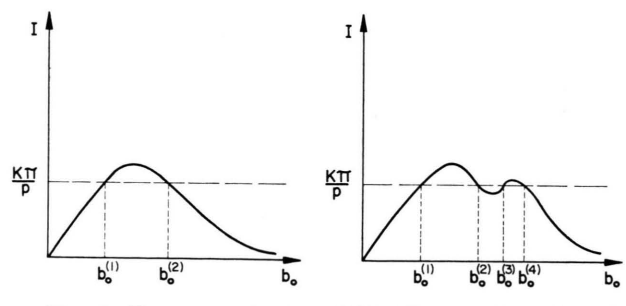

Fig. 2. Two examples for $I(b_0^2)$. The crossing of $\varkappa\pi/p$ with I determines the roots of equation (5.14). For sufficiently small values of p the straight line $\varkappa\pi/p$ is above $I(b_0^2)$ without intersection. For increasing p the straight line $\varkappa\pi/p$ is lowered. As the second example shows, solutions $b_0^{(i)}$, i. e. limit cycles, may also disappear with increasing p.

a definite threshold, i.e. $p>p_{\text{th}}$, two, respectively an even number, of solutions appear which we will denote by $b_0^{(i)}$, $i=1,2,\dots$.

Starting with one of the $b_0^{(i)}$, we can calculate at once further corrections of the amplitude and period of the solution under investigation. Taking into account again (4.3) we obtain from (5.8) after some calculation the first correction to the period:

$$\eta_1^{(i)} = -\frac{3}{2}\pi \frac{e^2 x_{12}^2 (b_0^{(i)})^2}{\hbar^2 \omega_0^2}; \quad i=1,2,\dots. \quad (5.16)$$

From (5.11) we can now determine $b_1^{(i)}$ and subsequently from (5.10) $\eta_2^{(i)}$, since the integrals occuring in these equations are only functions of $b_0^{(i)}$. However, as a further discussion is only possible, if the function $g(E^2)$ is known explicitly, we renounce here writing down the somewhat clumsy expressions for $b_1^{(i)}$ and $\eta_2^{(i)}$.

6. Conclusions

The principal consequence of the introduction of loss and source terms is the fact that instead of a set of periodic solutions, lying densely in the whole phase plane, we obtain a certain number of discrete periodic solutions. These bifurcate in pairs, when the pump parameter assumes certain critical values. As may be seen from Fig. 2, the number of these bifurcations is determined by the number of extrema of $g(E^2)$. Except at these critical parameter values our system is structurally stable, i.e. sufficiently small, but otherwise arbitrary perturbations of the differential equations cannot change the qualitative character of the totality of solutions.

It remains still to investigate the stability of these isolated periodic solutions. For this purpose we examine the behavior of solutions in the neighborhood of the limit cycles, i.e. we choose for initial conditions of the test solution $E=0$, $E'=\tilde{b}+b$. After one cycle in the phase plane, spanned by E and E', the variable E will again be zero, whereas E' assumes a value near $\tilde{b}$. The time θ necessary for this cycle can be given in implicit form by means of (4.8)

$$\tilde{b}\sin\theta + \mu\int_0^\theta \sin(\theta-s)\, F(E(s), E'(s))\, \mathrm{d}s = 0. \quad (6.1)$$

The sequence function Δ, i.e. the difference of the values of E before and after the cycle, reads

$$\Delta(\tilde{b},\mu) \equiv E'(\theta,\mu) - E'(0,\mu) = \tilde{b}(\cos\theta - 1) \quad (6.2)$$
$$+ \mu\int_0^\theta \cos(\theta-s)\, F(E(s), E'(s))\, \mathrm{d}s.$$

A simple geometric argument shows that the stability of periodic solutions can easily be examined by means of the sequence function $\varDelta$ (Poincaré's method). In the neighborhood of an unstable limit cycle all solutions tend away from it and thus we must have, evidently,

$$\begin{aligned} \varDelta(\tilde{b}) &< 0 \quad \text{for} \quad \tilde{b} < b, \\ \varDelta(b) &= 0, \\ \varDelta(\tilde{b}) &> 0 \quad \text{for} \quad \tilde{b} > b. \end{aligned} \tag{6.3}$$

This condition for the unstable character of a limit cycle may be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{b}} \varDelta(\tilde{b}) \Big|_{\tilde{b}=b} > 0 \tag{6.4}$$

in view of the continuous dependence of $\varDelta$ on $\tilde{b}$. The same argument leads for a stable limit cycle to

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{b}} \varDelta(\tilde{b}) \Big|_{\tilde{b}=b} < 0. \tag{6.5}$$

In order to connect these stability conditions with quantities introduced in the preceding section, we expand $\varDelta(\tilde{b}, \mu)$ in a power series with respect to μ. Taking into account

$$\theta = 2\pi + \theta_1 \mu + \cdots, \tag{6.6}$$

we obtain

$$\varDelta(\tilde{b}, \mu) = \mu \int_0^{2\pi} \cos s\, F(\tilde{b} \sin s, \tilde{b} \cos s)\, \mathrm{d}s + 0(\mu^2) \tag{6.7}$$

and the derivative

$$\frac{\mathrm{d}\varDelta}{\mathrm{d}\tilde{b}}\Big|_{\tilde{b}=b} = \mu \frac{\mathrm{d}}{\mathrm{d}\tilde{b}} \int_0^{2\pi} \cos s\, F(\tilde{b} \sin s, \tilde{b} \cos s)\, \mathrm{d}s + 0(\mu^2). \tag{6.8}$$

When we expand b as in (5.5) and employ the explicit form of F (4.7), we may write, finally,

$$\begin{aligned} \frac{\mathrm{d}\varDelta}{\mathrm{d}\tilde{b}}\Big|_{\tilde{b}=b} &= \mu \frac{\mathrm{d}}{\mathrm{d}b_0} \int_0^{2\pi} \cos s\, F(b_0 \sin s, b_0 \cos s)\, \mathrm{d}s + 0(\mu^2) \\ &= \mu \frac{\mathrm{d}}{\mathrm{d}b_0} I(b_0^2) + 0(\mu^2). \end{aligned} \tag{6.9}$$

Thus, for sufficiently small μ we see that the derivative of $I(b_0^2)$ decides on the stability of the limit cycle passing through $E = 0$, $E' = b \approx b_0$.

Now, we may conclude from Fig. 2 that the smallest limit cycle is always unstable, playing the part of a watershed which separates solutions tending towards zero and solutions tending towards the next limit cycle. The greatest limit cycle is always stable and possible limit cycles situated between show an alternating stability character. The phase portrait for the ensemble of solutions is indicated in Fig. 3.

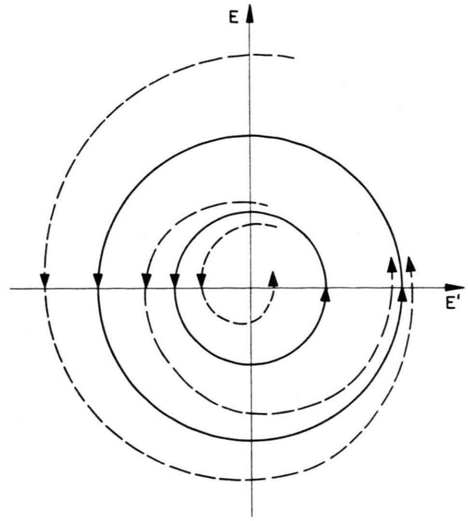

Fig. 3. Phase portrait for two limit cycles. All solutions beyond the outer limit cycle are bounded and will finally approach this cycle. Solutions with initial conditions within the inner limit cycle will tend to zero. The only physically realizable stationary solution is the outer limit cycle, apart from the zero solution.

Only the stable periodic solutions are realizable in practise. They are also the limit profiles towards which all other wave solutions tend. The frequency of these periodic solutions is given by

$$\begin{aligned} \frac{\omega_0}{\omega_i} &= 1 + \frac{1}{2\pi} \eta_1^{(i)} + \cdots \\ &= 1 - \frac{3}{4} \frac{e^2 x_{12}^2 (b_0^{(i)} 1)^2}{\hbar^2 \omega_0^2} + \cdots. \end{aligned} \tag{6.8}$$

Within the range of validity of the perturbation theory the frequencies ω_i are again always greater than ω_0. Since the $b_0^{(i)}$ are functions of the pump parameter p, the frequencies ω_i become also functions of p. From Fig. 2 it can be deduced that the frequencies of all stable isolated periodic solutions increase with increasing p, provided p does not assume a critical value for which our system loses its structural stability.

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